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Linear Control Theory Applied to a Minimum–Maximum Problem

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1. INTRODUCTION

The purpose of this paper is to use a linearization technique plus well-established linear control theory to derive relevant information concerning the extremal functions in a minimum–maximum problem. The problem in question is to minimize the expression

$$\operatorname*{ess\,sup}_{a<t<b} F(t, x(t), \dot{x}(t),..., x^{(n)}(t))$$

over the class of functions x which are "smooth" enough and satisfy suitable boundary conditions at a and b. Here, F is a given function in C^1 of $n+2$ variables. This is called an inf–sup or a minimum–maximum problem. One main result will be proved. It says, roughly, that if x_0 is a minimizing function satisfying a certain "nondegeneracy" condition, then $F(t, x_0(t), \dot{x}_0(t),..., x_0^{(n)}(t))$ is constant on (a, b). Also some further information is obtained, and it turns out that, for wide classes of functions F, the analysis is easily carried further to give more information, for instance on the "spline properties" of x_0. Examples of this are given in the last section.

This problem was treated by the present author in [2], using methods from nonlinear optimal control theory. The theorem proved here includes the main result in [2] as a corollary, as shown in Section 4. We are not aiming at a complete discussion of the extremum problem. A sketch of the background for these problems, including the spline concept, is given in [2]. This includes interesting theorems by Glaeser [3] for the case $F \equiv (x^{(n)}(t))^2$ and by McClure [6] for the case $F \equiv (x^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)\, x^{(k)}(t))^2$. The case $n=1$ has been studied from various aspects by the author; references are given in [2].

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2. Technical Preliminaries; Notations

We will study an extremum problem on a basic interval $a \leqslant t \leqslant b$, and will first define the class of admissible functions. Consider the Sobolev space

$$\begin{aligned} W^{n,\infty} &= W^{n,\infty}[a,b] \\ &= \{f \in R[a,b] \mid f^{(v)} \text{ is absolutely continuous for} \\ &\qquad v = 0, 1,..., n-1 \text{ and } \|f^{(n)}\|_{L^\infty} < \infty\}. \end{aligned}$$

If $X_0^{(v)}, X_1^{(v)}$ for $v = 1, 2,..., n$ are given real values, we consider

$$\begin{aligned} U = \{f \in W^{n,\infty} \mid f^{(v)}(a) = X_0^{(v+1)}, f^{(v)}(b) = X_1^{(v+1)} \\ \text{for } v = 0, 1,..., n-1\}. \end{aligned}$$

Then U will be the class of admissible functions. Further, let $F = F(t, y_0, y_1,..., y_n)$ be a given function in $C^1([a,b] \times R^{n+1})$. Then, for any $x \in W^{n,\infty}[a,b]$ the quantity

$$H(x) = \operatorname*{ess\,sup}_{a<t<b} F(t, x(t), \dot{x}(t),..., x^{(n)}(t))$$

is well defined. The problem is now to minimize the functional H over U. Naturally, other boundary conditions might also be of interest, but will not be used in this paper.

Let x be an arbitrary element in $W^{n,\infty}[a,b]$. Write $M = H(x)$ and consider the set

$$E_\epsilon = \{t \mid a < t < b, F(t, x(t), \dot{x}(t),..., x^{(n)}(t)) \geqslant M - \varepsilon\}.$$

Further, let $D_{k+2}F$ denote the partial derivative of F with respect to variable number $k+2$. We then have

Lemma 1. *Suppose that there is an $\varepsilon > 0$ and a $y \in W^{n,\infty}[a,b]$ so that*

$$\operatorname*{ess\,inf}_{t \in E_\varepsilon} \sum_{k=0}^{n} D_{k+2}F(t, x(t), \dot{x}(t),...)\, y^{(k)}(t) > 0. \tag{1}$$

Then $H(x - \lambda y) < H(x)$ for all sufficiently small $\lambda > 0$.

Proof. Clearly,

$$\begin{aligned} &F(t, x(t) - \lambda y(t),...) \\ &\qquad = F(t, x(t),...) - \lambda \cdot \sum_{k=0}^{n} D_{k+2}F(t, x(t),...)\, y^{(k)}(t) + R(t, \lambda), \end{aligned}$$

where $R(t, \lambda) = o(\lambda)$, uniformly in t. Thus, for $t \in E_\varepsilon \backslash$(a null set)

$$F(t, x(t) - \lambda y(t),...) \leqslant M - \theta \cdot \lambda + R(t, \lambda),$$

where $\theta = \text{ess inf} \sum_k \cdots > 0$. Further, $|R(t, \lambda)| \leqslant (\theta/2)\lambda$ for $0 < \lambda < \lambda_1$ and then $F(t, x - \lambda y,...) \leqslant M - (\theta/2)\lambda$. For $t \notin E_\varepsilon$ we have

$$\lim_{\lambda \to 0} F(t, x - \lambda y,...) = F(t, x,...) < M - \varepsilon$$

with uniform convergence.

The lemma follows easily from this.

LEMMA 2. *Assume that the vector funtion $\eta(t) = (\eta_1(t),..., \eta_n(t)) \neq 0$ satisfies a linear system of the form*

$$\begin{aligned}
\dot{\eta}_1 &= \qquad\quad \psi_1(t)\, \eta_n, \\
\dot{\eta}_2 &= -\eta_1 \quad + \psi_2(t)\, \eta_n, \\
\dot{\eta}_3 &= -\eta_2 \quad + \psi_3(t)\, \eta_n, \\
&\ \vdots \\
\dot{\eta}_n &= -\eta_{n-1} + \psi_n(t)\, \eta_n,
\end{aligned}$$

a.e. on an interval (a, b). The functions ψ_k are in $L^\infty(a, b)$. Then η_n can have at most a finite number of zeros on (a, b). (It is understood that η is absolutely continuous.)

Proof. Consider $\eta(t)$ as a column vector and write the system as

$$\dot{\eta} = -A_0 \eta + \psi(t)\, \eta_n,$$

where

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Assume that η_n has an infinity of zeros on $[a, b]$. After reversing the t-axis we may write

$$\dot{\eta} = A_0 \eta + \varphi(t)\, \eta_n,$$

where $\varphi(t) \in L^\infty$ is an $(n \times 1)$ vector function. We can assume that the zeros of η_n cluster at $t=0$. We have

$$\eta(t) = e^{A_0 t}\eta(0) + \int_0^t e^{A_0(t-s)}\varphi(s)\,\eta_n(s)\,ds.$$

Now the matrix A_0 is nilpotent and $e^{A_0 t}$ is easily computed. In fact, we have (see [4, p. 99])

$$e^{A_0 t} = \begin{bmatrix} 1 & 0 & 0 & \cdots & & \\ t & 1 & 0 & \cdots & & \\ t^2/2 & t & 1 & \cdots & & \\ \vdots & \vdots & \vdots & & & \\ \dfrac{t^{n-1}}{(n-1)!} & \dfrac{t^{n-2}}{(n-2)!} & \dfrac{t^{n-3}}{(n-3)!} & \cdots & \dfrac{t^2}{2} & t & 1 \end{bmatrix}.$$

By taking the nth component in the above equation for $\eta(t)$ we find

$$\eta_n(t) = P(t) + \int_0^t (e^{A_0(t-s)}\varphi(s))_n\,\eta_n(s)\,ds, \tag{2}$$

where the polynomial $P(t) \not\equiv 0$, since $\eta(0) \neq 0$. Let $c_k t^k$ be the lowest order term in $P(t)$. Thus $|\eta_n(t)| \leqslant c'_k |t|^k + d_k |\int_0^t| \eta_n(s)\,|ds|$ for some constants $c'_k > 0$ and d_k. From the generalized Gronwall inequality ([4, p 36]) we infer that $\eta_n(t) = O(t^k)$. But then the integral in (2) is $O(t^{k+1})$. Hence it follows from (2) and our choice of k that the zeros of η_n cannot cluster at $t=0$.

The contradiction completes the proof.

3. The Main Result

Because of the boundary conditions, the function y in Lemma 1 should satisfy $y^{(k)}(a) = y^{(k)}(b) = 0$ for $k = 0, 1,..., n-1$, in order to be useful. Put $F_k(t) = D_{k+2}F(t, x(t),..., x^{(n)}(t))$. Clearly, $F_k \in L^\infty$ for any $x \in W^{n,\infty}$. We thus have a differential equation

$$\sum_{k=0}^{n} F_k(t)\,y^{(k)}(t) = \omega(t),$$

where it is crucial that $\omega(t) \geqslant \delta > 0$ for $t \in E_\varepsilon$, for some positive ε and δ. It is natural and expected that the leading coefficient F_n is important in the further analysis. The main result is

THEOREM. *Suppose that H takes on its minimum over the class U at x_0. Suppose that there exist ε and μ, both positive, such that*

$$|F_n(t)| \geqslant \mu \qquad \text{a.e. on } E_\varepsilon.$$

Then $F(t, x_0(t), \dot{x}_0(t), \dots, x_0^{(n)}(t)) = M$ a.e. on (a, b). Further, the interval (a, b) can be divided into a finite number of subintervals $\{J_k\}_1^N$ such that, for each J_k, either $F_n(t) \geqslant \mu$ a.e. on J_k or else $F_n(t) \leqslant -\mu$ a.e. on J_k.

Proof. As an attempt, put $\sum_{k=0}^{n} F_k(t)\, y^{(k)}(t) = u(t)$ on E_ε and $y^{(n)}(t) = w(t)$ on $[a, b]\backslash E_\varepsilon$, except for null sets. Now y is to be constructed by appropriate choice of u and w. First, write

$$y^{(n)}(t) = \sum_{k=0}^{n-1} \left(-\frac{F_k(t)}{F_n(t)}\right) y^{(k)}(t) + \frac{u(t)}{F_n(t)}$$

on E_ε.

We introduce functions $\{G_k(t)\}_{k=0}^{n-1}$, $b(t)$, and $d(t)$ in L^∞ by putting

$$\begin{aligned} G_k(t) &= -F_k(t)/F_n(t), && \text{for } t \in E_\varepsilon, \\ &= 0, && \text{for } t \notin E_\varepsilon; \end{aligned}$$

$$\begin{aligned} b(t) &= \frac{1}{F_n(t)}, && t \in E_\varepsilon, \\ &= 0, && t \notin E_\varepsilon; \end{aligned} \qquad \text{and} \qquad \begin{aligned} d(t) &= 0, && t \in E_\varepsilon, \\ &= 1, && t \notin E_\varepsilon. \end{aligned}$$

Then the attempt is summarized by writing

$$y^{(n)}(t) = \sum_{k=0}^{n-1} G_k(t)\, y^{(k)}(t) + b(t)u + d(t)w.$$

Transform this equation into a first-order system by writing $x_1 = y$, $x_2 = \dot{y}, \dots$, $x_n = y^{(n-1)}$. The system is then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= \sum_{k=1}^{n} G_{k-1}(t)\, x_k + b(t)u + d(t)w. \end{aligned}$$

Introduce matrix functions in L^∞

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ G_0 & G_1 & G_2 & G_3 & \cdots & G_{n-1}(t) \end{bmatrix},$$

$$B(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b(t) \end{bmatrix}, \quad \text{and} \quad D(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ d(t) \end{bmatrix}.$$

In matrix form, the system is written as

$$\dot{x} = A(t)x + B(t)u + D(t)w. \tag{*}$$

It is convenient to consider this as a linear control system in R^n, the control variables being u and w. It fits perfectly into the machinery of [5, Chap. 2, p. 68].

It is required to steer the system from $x(a) = 0$ to $x(b) = 0$ by using control functions u and w so that $\operatorname{ess\,inf} u(t) > 0$. The solution of (*) for $t = b$ is

$$x(b) = \Phi(b) \int_a^b \Phi(s)^{-1} [B(s)\,u(s) + D(s)\,w(s)]\,ds,$$

where $\Phi(s)$ is a fundamental matrix for $\dot{x} = A(t)x$. It is assumed here that $x(a) = 0$. Since $\Phi(b)$ is nonsingular, it is required to find u and w so that $\int_a^b \Phi(s)^{-1} [Bu + Dw]\,ds = 0$ and $\operatorname{ess\,inf} u(t) > 0$.

Let us try $u(t) \equiv 1$. If the equation

$$\int_a^b \Phi(s)^{-1} D(s)\,w(s)\,ds = -\int_a^b \Phi(s)^{-1} B(s)\,ds = X$$

has a solution $w \in L^\infty$, then this will contradict the optimality of the function x_0 in the theorem. Assume now that $[a, b] \setminus E_\varepsilon$ is a set of positive measure. We claim that the equation

$$L_0(w) = \int_a^b \Phi(s)^{-1} D(s)\,w(s)\,ds = X$$

has a solution. If this is not true, then the image of L^∞ under the mapping L_0 is a proper subspace of R^n, and then there is a normal $N \neq 0$ such that

$$\int_a^b N\Phi(s)^{-1} D(s)\, w(s)\, ds = 0 \qquad \text{for all} \quad w \in L^\infty.$$

Put $N\Phi(s)^{-1} = \eta(s) = (\eta_1(s),..., \eta_n(s))$. From the definition of $D(t)$ it follows that

$$\int_{[a,b]\setminus E_\varepsilon} \eta_n(s)\, w(s)\, ds = 0 \qquad \text{for all} \quad w \in L^\infty.$$

Consequently, $\eta_n(s) = 0$ a.e. on $[a, b]\setminus E_\varepsilon$, i.e., on a set of *positive measure*. But it is well known that $\eta(s) = N\Phi(s)^{-1} \neq 0$ satisfies the adjoint system $\dot{\eta} = -\eta A(t)$.

From the form of the matrix $A(t)$ it is clear that Lemma 2 is applicable and, consequently, η_n can only have a finite number of zeros. But this gives a contradiction and therefore the assumption that $[a, b]\setminus E_\varepsilon$ has positive measure is wrong. Thus, E_ε has full measure.

Clearly, this must be true for all sufficiently small $\varepsilon > 0$. It follows that $F(t, x_0(t),..., x_0^{(n)}(t)) = M$ a.e. on (a, b).

Now the variable w has finished its role and it remains to study the equation $L(u) = \int_a^b \Phi(s)^{-1} B(s)\, u(s)\, ds = 0$ under the conditions $u \in L^\infty$ and $\operatorname{ess\,inf} u > 0$. Let U^+ denote the class of all such control functions. Since x_0 is optimal, it follows from the analysis that $0 \notin L(U^+)$. Consider the set $L(U^+)$. Clearly, it is a convex cone in R^n (in the terminology of [7, p. 13]).

Since $0 \notin L(U^+)$, it follows (see [7, p. 101]) that there is a vector $N \neq 0$ in R^n such that $N \cdot L(u) \geqslant 0$ for all $u \in U^+$. Thus, $\int_a^b N\Phi(s)^{-1} B(s)\, u(s)\, ds \geqslant 0$ for all $u \in U^+$. Again, write $N\Phi(s)^{-1} = \eta(s) = (\eta_1(s),..., \eta_n(s))$. As above, we know that η satisfies $\dot{\eta} = -\eta A(t)$, and, again by Lemma 2, we know that η_n has only a finite number of zeros.

From the definition of $B(s)$ it follows that

$$\int_a^b \frac{\eta_n(s)}{F_n(s)}\, u(s)\, ds \geqslant 0 \qquad \text{for all} \quad u \in U^+.$$

Obviously, $\eta_n(s)/F_n(s) \geqslant 0$ a.e. on $[a, b]$. Thus, sign $F_n(s) = \text{sign}\, \eta_n(s)$ a.e., and the theorem follows.

Remark. The concept of an absolutely minimizing function was introduced in [1, p. 45]. The idea is that the function in question must solve the extremum problem, not only on the given interval, but also on each subinterval, the boundary data then being given by the function itself. Note that in the classical calculus of variations every minimizing function is absolutely minimizing, whereas this is no longer true for extremum problems of the present type.

COROLLARY. *The function* x_0 *in the above theorem is absolutely minimizing.*

Proof. If this is not true, then there is a subinterval $[\alpha, \beta] \subset [a, b]$ and a function $x_1 \in W^{n,\infty}[\alpha, \beta]$ such that $x_1^{(k)}(\alpha) = x_0^{(k)}(\alpha)$ and $x_1^{(k)}(\beta) = x_0^{(k)}(\beta)$ for $k = 0, 1, \dots, n-1$, and such that

$$\operatorname*{ess\,sup}_{\alpha < t < \beta} F(t, x_1(t), \dots, x_1^{(n)}(t)) < M.$$

Assume for instance that $a < \alpha < \beta < b$. Consider as in the proof of the theorem the control system $\dot{x} = A(t)x + B(t)u$, now over the two intervals $[a, \alpha]$ and $[\beta, b]$. Use the control $u(t) \equiv 1$ and zero data at $t = a$ to obtain a function $y(t)$ on $[a, \alpha]$. Similarly, use $u(t) \equiv 1$ and zero data at $t = b$ to obtain $z(t)$ on $[\beta, b]$. Let $\lambda > 0$ be a parameter at our disposal.

Consider $Y_\lambda(t) = x_0(t) - \lambda y(t)$ on $[a, \alpha]$ and $Z_\lambda(t) = x_0(t) - \lambda z(t)$ on $[\beta, b]$. From Lemma 1 we know that $H(Y_\lambda) < M$ and $H(Z_\lambda) < M$, for λ small enough. Choose a function $w \in W^{n,\infty}[\alpha, \beta]$ such that $w^{(k)}(\alpha) = y^{(k)}(\alpha)$ and $w^{(k)}(\beta) = z^{(k)}(\beta)$ for $k = 0, 1, \dots, n-1$. Form the function $X \in W_0^{n,\infty}[a, b]$ defined by

$$\begin{aligned} X(t) &= y(t), && \text{for } a \leqslant t \leqslant \alpha, \\ &= w(t), && \text{for } \alpha \leqslant t \leqslant \beta, \\ &= z(t), && \text{for } \beta \leqslant t \leqslant b. \end{aligned}$$

Also, define $x_2 \in W^{n,\infty}[a, b]$ by

$$\begin{aligned} x_2(t) &= x_0(t), && \text{for } a \leqslant t \leqslant \alpha, \\ &= x_1(t), && \text{for } \alpha \leqslant t \leqslant \beta, \\ &= x_0(t), && \text{for } \beta \leqslant t \leqslant a. \end{aligned}$$

It is obvious that $H(x_2 - \lambda X) < H(x_0) = M$ if $\lambda > 0$ is small enough. This contradicts the optimality of x_0 and completes the proof.

Further remarks. A function x can be absolutely minimizing without satisfying the conditions of the theorem. An example of this is given in [1, p. 53]. The same example shows that the assumption $|F_n(t)| \geqslant \mu$ on E_ε cannot be omitted. See also [2, p. 146].

4. Some Particular Cases of Interest

We shall discuss some applications of the above theorem.

A. *Comparison with* [2]

Let us make an assumption concerning the function $F=F(t, y_0, y_1, \ldots, y_n)$ which was made in [2, p. 145], namely, there is a function $\omega \in C([a, b] \times R^n)$ so that

$$\begin{aligned} \partial F/\partial y_n \text{ is } &> 0, && \text{if } \ y_n > \omega(t, y_0, y_1, \ldots, y_{n-1}), \\ &= 0, && \text{if } \ y_n = \omega(\cdots), \\ &< 0, && \text{if } \ y_n < \omega(\cdots). \end{aligned}$$

Introduce the "minimum function" $m(t, y_0, y_1, \ldots, y_{n-1}) = F(t, y_0, y_1, \ldots, y_{n-1}, \omega(t, y_0, y_1, \ldots, y_{n-1}))$. Let $x_0 \in W^{n,\infty}$ be a minimizing function such that

$$m(t, x_0(t), \dot{x}_0(t), \ldots, x_0^{(n-1)}(t)) < M$$

holds for $a \leqslant t \leqslant b$. This is condition (*) in [2, p. 145]. By uniform continuity there exist ε and δ_1, both positive and independent of $t \in [a, b]$ such that $F(t, x_0(t), \ldots, x_0^{(n-1)}(t), Z) < M - \varepsilon$ if $|Z - \omega(t, x_0(t), \ldots, x_0^{(n-1)}(t))| \leqslant \delta_1$. Further, by continuity and the condition on $\partial F/\partial y_n$, there exists a $\delta_2 > 0$, not depending on t such that

$$\left| \frac{\partial F}{\partial y_n}(t, x_0(t), \ldots, x_0^{(n-1)}(t), Z) \right| > \delta_2$$

if

$$|Z - \omega(t, x_0(t), \ldots, x_0^{(n-1)}(t))| \geqslant \delta_1$$

and

$$|Z| \leqslant \|x_0^{(n)}\|_{L^\infty}.$$

But this obviously implies that

$$\left| \frac{\partial F}{\partial y_n}(t, x_0(t), \ldots, x_0^{(n)}(t)) \right| > \delta_2$$

on the set E_ε. Consequently, our present theorem is applicable.

Thus $F(t, x_0(t), \ldots, x_0^{(n)}(t)) = M$ a.e. and (a, b) can be divided into open subintervals $\{J_k\}_1^N$ so that $(\partial F/\partial y_n)(t, x_0(t),..., x_0^{(n)}(t))$ has fixed sign on each J_k (possibly apart from a null set).

Consider an interval J_k where, let us say, $(\partial F/\partial y_n)(\cdots) > 0$. Thus $x_0^{(n)}(t) > \omega(t, x_0(t),..., x_0^{(n-1)}(t))$ a.e. on J_k. But clearly the equation

$$F(t, x_0(t),..., x_0^{(n-1)}(t), Z) = M$$

has a unique solution $Z > \omega(t,...)$ for *every* $t \in \bar{J}_k$. Furthermore, this solution can be represented as $Z = \psi(t, x_0(t),..., x_0^{(n-1)}(t))$, where $\psi \in C^1$.

Thus $x_0^{(n)}(t) = \psi(t, x_0(t),..., x_0^{(n-1)}(t))$ on *all* of J_k. It follows immediately that $x_0 \in C^{n+1}(J_k)$. Finally, it is clear that $x_0^{(n)}(t)$ has a jump discontinuity at every point where $\partial F/\partial y_n$ changes sign, i.e., at a finite number of points.

We thus obtain the theorem in [2] as a corollary of the present theorem.

B. *Comparison with Glaeser's Case*

Let $F(t, x(t),..., x^{(n)}(t)) \equiv \psi(t)(x^{(n)}(t))^2$, where $\psi(t) > 0$ and $\psi \in C^1[a, b]$. Let x_0 be a minimizing function (which always exists here) and put $M = H(x_0)$. If $M = 0$, then obviously x_0 is a polynomial of degree $\leqslant n - 1$ and the solution is unique.

Let $M > 0$. Now $F_n(t) = 2\psi(t)\, x_0^{(n)}(t)$ and the present theorem is obviously applicable. Thus $x_0^{(n)}(t) = \pm\sqrt{M/\psi(t)}$, with a finite number of sign changes. How many intervals J_k can there be? Clearly the system satisfied by $\eta(t) = (\eta_1(t),..., \eta_n(t)) \neq 0$ in the end of the proof is simply

$$\begin{aligned}
\dot{\eta}_1 &= 0,\\
\dot{\eta}_2 &= -\eta_1,\\
\dot{\eta}_3 &= -\eta_2,\\
&\;\vdots\\
\dot{\eta}_n &= -\eta_{n-1},
\end{aligned}$$

and thus η_n has at most $(n - 1)$ zeros. Thus the number of intervals J_k is at most n.

This should be compared with two theorems by Glaeser [3], also stated in [2, pp. 142–143]. We see that some of Glaeser's statements easily carry over to the more general case.

C. *Comparison with McClure's Case*

Consider the case

$$F \equiv \left(x^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)\, x^{(k)}(t)\right)^2,$$

where each $a_k \in C^k[a, b]$. The existence of a minimizing function x_0 is clear. Put $M = H(x_0)$. If $M = 0$, then x_0 satisfies

$$x_0^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)\, x^{(k)}(t) = 0,$$

together with all given boundary data.

Let $M > 0$. Obviously the condition "$|F_n(t)| \geqslant \mu$ on E_ε" is satisfied and so our theorem can be applied. Thus,

$$x_0^{(n)}(t) = -\sum_{k=0}^{n-1} a_k(t)\, x_0^{(k)}(t) \pm \sqrt{M},$$

with a finite number of switches. Hence x_0 is a "perfect A-spline" with

$$A \equiv D^n + \sum_{k=0}^{n-1} a_k(t)\, D^k$$

(see [2, p. 143]). The switches coincide with sign changes of some nontrivial solution Φ of the adjoint equation $A^*\Phi = 0$.

This gives McClure's Theorem C again [2, p. 143], except for the uniqueness of x_0.

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